Oct 24, 2022

Week 8

2020 A Adv. Cal. II

Line Integrals

A <u>parametric curve</u> is a map : $[a,b] \to \mathbb{R}^2$, or $\mathbb{R}^3$,

$$\vec{c}(t) = (g(t), h(t), j(t))$$
$$= g(t)\,\hat{i} + h(t)\,\hat{j} + j(t)\,\hat{k} \quad \text{or}$$
$$= (x(t), y(t), z(t)) \quad \text{or}$$
$$= x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k} \quad \text{or}$$

so that all components are continuous. (When $n=2$, drop the third component.) It is <u>regular</u> if all components are continuously differentiable on (a,b) and $\vec{c}'(t) \neq (0,0,0)$, ie $|\vec{c}'(t)| > 0$ on (a,b).

For a regular curve $\vec{c}$,

$\vec{c}'(t)$ — velocity of $\vec{c}$ at $\vec{c}(t)$,

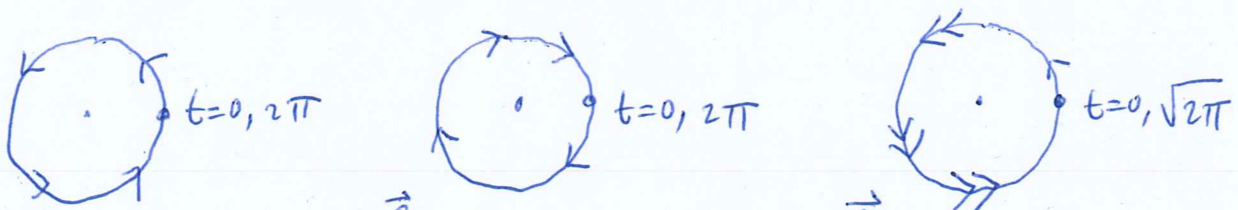
$|\vec{c}'(t)|$ — speed of $\vec{c}$ at $\vec{c}(t)$,

$$\hat{t} = \frac{\vec{c}'(t)}{|\vec{c}'(t)|} \quad \text{— tangent of } \vec{c} \text{ at } \vec{c}(t)$$

e.g. $\vec{c}_1(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $t \in [0, 2\pi]$

$\vec{c}_2(t) = \cos t\,\hat{i} - \sin t\,\hat{j}$, $t \in [0, 2\pi]$

the image of $\vec{c}_1, \vec{c}_2$ are the same, the unit circle, but as t increases, $\vec{c}_1(t)$ runs anticlockwise from $(1,0)$ back to $(1,0)$ at $t = 2\pi$, while $\vec{c}_2(t)$ runs clockwise.



$\vec{c}_1$ $\vec{c}_2$ $\vec{c}_3$

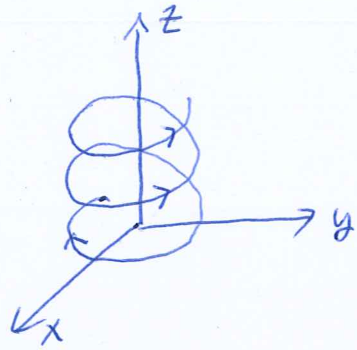
$\vec{C}_3(t) = (\cos t^2, \sin t^2)$, $t \in [0, \sqrt{2\pi}]$

$|C_1'(t)| = 1$, $|C_2'(t)| = 1$, $|C_3'(t)| = 2t$, so C_3 runs at non-constant speed.

e.g. Helix, $\vec{\gamma}(t) = (r\cos t, r\sin t, t)$, $t \in [0, 2\pi]$.

$\vec{\gamma}'(t) = (-r\sin t, r\cos t, 1)$

$|\vec{\gamma}'(t)| = \sqrt{1+r^2}$ constant speed



Let $\vec{C}(t)$, $t \in [a,b]$, be a regular parametric curve.

Let $C = \{\vec{C}(t) : t \in [a,b]\}$ be its image.

Let f be a continuous function defined on C.

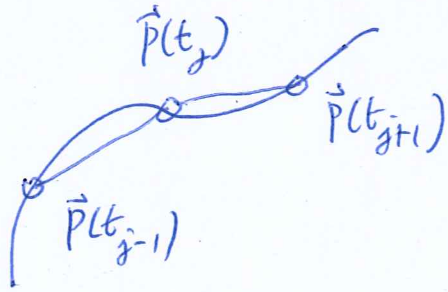
Want to define $\int_{\vec{C}} f\, ds$.

Motivation: Image f is the density of the wire C.

$\int_C f\, ds$ should give the mass of the wire.

Let $a = t_0 < t_1 < t_2 < \dots < t_n = b$ be a partition in $[a, b]$.

Let $\vec{P}_j = \vec{C}(t_j)$.



the approximate mass is

$$\sum_{j=1}^{n} f(\vec{C}(t_j'))\,|\vec{P}_j - \vec{P}_{j-1}|,$$

t_j' tag point.

$|\vec{P}_j - \vec{P}_{j-1}|$ is the line of the line segment bet. $\vec{P}_j$ and $\vec{P}_{j-1}$.

Now

$$|\vec{P}_j - \vec{P}_{j-1}| = |\vec{c}(t_j) - \vec{c}(t_{j-1})|$$

$$= |x(t_j)\hat{i} + y(t_j)\hat{j} - x(t_{j-1})\hat{i} - y(t_{j-1})\hat{j}|$$

$$= |(x(t_j) - x(t_{j-1}))\hat{i} + (y(t_j) - y(t_{j-1}))\hat{j}|$$

$$= |x'(t_j^*)\Delta t_j \hat{i} + y'(t_j^{**})\Delta t_j \hat{j}|$$ (mean-value theorem)

$$= \sqrt{x'(t_j^*)^2 \Delta t_j^2 + y'(t_j^{**})^2 \Delta t_j^2}$$

$$= \sqrt{x'^2(t_j^*) + y'^2(t_j^{**})}\ \Delta t_j$$

$$\sim \sqrt{x'^2(t_{j-1}) + y'^2(t_{j-1})}\ \Delta t_j$$ ($\because t_j^{**}, t_j^*$ close to t_{j-1})

So, approximate mass

$$\sim \sum f(\vec{c}(t_j'))\sqrt{x'^2(t_{j-1}) + y'^2(t_{j-1})}\ \Delta t_j$$

$$\sim \sum f(\vec{c}(t_{j-1}))\sqrt{x'^2(t_{j-1}) + y'^2(t_{j-1})}\ \Delta t_j,$$ (t_j' close to t_{j-1})

which is a Riemann sum for the function

$$f(\vec{c}(t))\,|\vec{c}\,'(t)|.$$

Letting $\|P\| \to 0$, the Rieman sum tends to

$$\int_a^b f(\vec{c}(t))|\vec{c}\,'(t)|\,dt. \qquad (*)$$ good for $n = 2, 3$

So, we define the <u>line integral</u> of f <u>along</u> $\vec{c}$ to be $(*)$.
When $f \geq 0$, it is the mass of the wire with density f. When $f \equiv 1$, it gives the length of C.

e.g. Find the length of the curve

$$\vec{c}_1(t) = r\cos t\,\hat{i} + r\sin t\,\hat{j}, \quad t \in [0, 2\pi].$$

$$\vec{c}_2(t) = r\cos t^2\,\hat{i} + r\sin t^2\,\hat{j}, \quad t \in [0, \sqrt{2\pi}].$$

$$\vec{c}_3(x) = x\,\hat{i} + \sqrt{r^2 - x^2}\,\hat{j}, \quad x \in [-r, r].$$

$|\vec{c}_1'(t)| = r,$

$$\therefore \text{length} = \int_0^{2\pi} |c_1'(t)|\,dt = \int_0^{2\pi} r\,dt = 2\pi r.$$

$|\vec{c}_2'(t)| = 2tr$

$$\text{length} = \int_0^{\sqrt{2\pi}} |\vec{c}_2'(t)|\,dt = 2r\int_0^{\sqrt{2\pi}} t\,dt = 2\pi r.$$

$$|\vec{c}_3'(x)| = \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2} = \frac{r}{\sqrt{r^2 - x^2}}.$$

$\therefore$ length of (half circle)

$$= \int_{-r}^{r} |\vec{c}_3'(x)|\,dx = 2\int_0^r \frac{r\,dx}{\sqrt{r^2 - x^2}} \qquad x = r\sin\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{r^2\cos\theta\,d\theta}{\sqrt{r^2 - r^2\sin^2\theta}}$$

$$= 2\int_0^{\frac{\pi}{2}} r\,d\theta$$

$$= \pi r \quad \#$$

e.g. Find the length of the helix $\vec{\gamma}(t) = (\cos t, \sin t, t)$.

$\vec{\gamma}'(t) = (-\sin t, \cos t, t)$, $|\vec{\gamma}'(t)| = \sqrt{2}$.

$$\therefore \text{length} = \int_0^{2\pi} |\vec{\gamma}'(t)|\,dt = \sqrt{2} \times 2\pi \quad \#$$